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EXTENSION THEOREMS FOR SOME CLASSES OF CONTINUA

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It is proved that every mapping from a proper subcontinuum of a hereditarily unicoherent continuum X onto the Knaster's indecomposable continuum (onto a cone over a zerodimensional compact metric set) can be extended to a mapping defined on X.

Similarly, every mapping from a proper subcontinuum of a hereditarily indecomposable continuum onto a pseudoarc can be extended to a mapping defined on the whole space.

Both of the above results are generalizations of the author's earlier results to the nonmetric case. As a consequence it is obtained that a pseudoarc is continuously n-homogeneous.

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hereditarily unicoherent continuum	pseudoarc
hereditarily indecomposable continuum	extension of a mapping

Introduction

If X is a topological space and $x \in A \subset X$, then $\bar{A}$, int A, $K(x, A)$ denote respectively the closure of A in X, the interior of A in X and a component of x in A. All mappings considered in this paper are continuous. A continuum means a compact connected Hausdorff space.

The following is known (see [2], p. 152).

Proposition 1. *For every pair A and B of separated F_σ-sets in a normal space X, there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

1. Hereditarily unicoherent continua

We consider a continuum D as the limit of the inverse sequences $\{I_n, h_n\}$, where for each $n \geq 1$ we have $I_n = [0, 1]$, $h_n = h$ and h is given by

$$h(t) = \begin{cases} 3t & \text{for } 0 \leq t \leq \frac{1}{3}, \\ & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ -3t+3 & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases}$$

We denote the projection from the inverse limit D onto I_n by α_n. The following is known (see [4]).

Proposition 2. *The continuum D is chainable and indecomposable.*

Now, we will prove

Theorem 3. *Let E be a subcontinuum of a hereditarily unicoherent continuum X. If f is a continuous mapping from E onto D, then there is a continuous mapping f^* from X onto D such that $f^*|E = f$.*

Proof. For $n = 1, 2, \ldots$ put $A_n = (\alpha_{n+1} f)^{-1}([0, \frac{1}{3}))$, $B_n = (\alpha_{n+1} f)^{-1}((\frac{2}{3}, 1])$ and $C_n = (\alpha_{n+1} f)^{-1}((\frac{1}{3}, \frac{2}{3}))$. We will construct a sequence of continuous mappings $f_n : X \to I_n$ which satisfies the conditions

(1) $C_n \subset \operatorname{int} f_n^{-1}(1)$ and $C_{n+1} \cup C_{n+2} \cup \cdots \subset \operatorname{int} f_n^{-1}(0)$,

(2) $f_n|E = \alpha_n f$,

(3) $f_{n+1} = h f_n$ for $n > 1$.

We proceed by induction. There exist open sets $G, H \subset X$ such that $A_1 \cup B_1 \subset G$, $C_1 \subset H$ and $G \cap H = \emptyset$ by Proposition 1, because the sets $A_1 \cup B_1$ and C_1 are separated F_σ-sets. Similarly, there exist open sets $U, V \subset X$ such that $C_2 \cup C_3 \cup \cdots \subset U$, $(\alpha_1 f)^{-1}((0, 1]) \subset V$ and $U \cap V = \emptyset$), and there exist open sets U' and V' such that $C_1 \subset V'$, $C_2 \cup C_3 \cup \cdots \subset U'$ and $\bar{V}' \cap \bar{U}' = \emptyset$. Observe that the mapping

$$g_1(x) = \begin{cases} \alpha_1 f(x) & \text{for } x \in E, \\ 1 & \text{for } x \in \overline{H \cap V \cap V'}, \\ 0 & \text{for } x \in \overline{G \cap U \cap U'} \end{cases}$$

is well defined. It follows from Tietze's Extension Theorem that there is a continuous mapping $f_1 : X \to I_1$ which is an extension of g_1. One can easily check that f_1 satisfies (1) and (2) for $n = 1$.

Assume the functions $f_1, f_2, \ldots, f_n$ have been constructed in such a way that (1), (2) and (3) hold for $k \leq n$. Now, we will construct f_{n+1} by induction. We have obviously

(4) $\overline{A_n} \cup \overline{B_n} \subset X \setminus \operatorname{int} f_n^{-1}(1)$.

Consider the sets $M = (X \setminus \operatorname{int} f_n^{-1}(1)) \cup (\alpha_{n+1} f)^{-1}(\frac{1}{3}) \cup (\alpha_{n+1} f)^{-1}(\frac{2}{3})$. $A'_n = \overline{A_n} \cup (\alpha_{n+1} f)^{-1}(\frac{1}{3})$ and $B'_n = \overline{B_n} \cup (\alpha_{n+1} f)^{-1}(\frac{2}{3})$. It follows from (4) that $A'_n \cup B'_n \subset M$.

Moreover, the hereditary unicoherence of X implies that M is not connected between A'_n and B'_n. Therefore there is a decomposition of $M = P \cup Q$ into two closed disjoint sets P and Q such that $A'_n \subset P$ and $B'_n \subset Q$. Put $R = \operatorname{int} f_n^{-1}(1)$ and consider the mapping g_{n+1} defined on the set $R \cap (P \cup Q \cup E)$ by

$$g_{n+1}(x) = \begin{cases} \alpha_{n+1} f(x) & \text{for } x \in R \cap E, \\ \frac{1}{3} & \text{for } x \in R \cap P, \\ \frac{2}{3} & \text{for } x \in R \cap Q. \end{cases}$$

It follows from Tietze's extension theorem that there is an extension g^*_{n+1} of g_{n+1} which is defined on R and $g^*_{n+1}(R)=[\frac{1}{3},\frac{2}{3}]$. Let us define f_{n+1} by

$$f_{n+1}(x)=\begin{cases}\frac{1}{3}\cdot f_n(x) & \text{for } x\in P,\\ g^*_{n+1}(x) & \text{for } x\in R,\\ 1-\frac{1}{3}\cdot f_n(x) & \text{for } x\in Q.\end{cases}$$

One can easily verify that f_{n+1} satisfies the required conditions.

The functions f_n induce the surjection f^* from X onto D by (3) and then $f^*|E=f$ by (2). The proof of Theorem 3 is complete.

The omitted details in the above proof are similar to those from the proof of Theorem 2 in [3].

From Theorem 2 we infer

Corollary 4. *If the continuum D is contained in a hereditarily unicoherent continuum X, then it is a retract of X.*

Using only Proposition 1 from this paper instead of Proposition 2 in [3], one can observe that the proof of Theorem 3 in [3] gives us the following

Theorem 5. *Let F be a cone over a zerodimensional compact metric set. If E is a subcontinuum of a hereditarily unicoherent continuum X and f is a continuous mapping from E onto F, then there is a continuous mapping f^* from X onto F such that $f^*|E=f$.*

Note that Theorems 3 and 5 solve the problem raised in [3], p. 183.

2. Hereditarily indecomposable continua

Now, we will extend results from [4] to the nonmetric case. As above, we will omit details of proofs which are similar to those from [4] and we will give only the main ideas in constructions.

Put $I=[0,1]$ and let R denote the set of rational numbers from I.

We say that (see [1]) a mapping $h:I\to I$ is an N-mapping if h satisfies the following conditions:

(i) $h(q)$ is rational if and only if q is rational,

(ii) there exist four rationals a, b, c, d with $0<a<c<1$ and $0<d<b<1$ and $h(a)=b$ and $h(c)=d$,

(iii) $h(0)=0$ and $h(1)=1$,

(iv) each of $h|[0,a]$, $h|[a,c]$ and $h|[c,1]$ is a homeomorphism.

If $f:X\to I$ and $t\in(0,1)$, then $f^-(t)=f^{-1}([0,t))$, $f^+(t)=f^{-1}((t,1])$, $f^<(t)=f^{-1}([0,t])$ and $f^>(t)=f^{-1}([t,1])$. We say that a function f from X onto I is separating if $\overline{f^-(t)}\cap\overline{f^+(t)}=\emptyset$ for $t\in R\setminus\{0,1\}$. The following is known (see [4], Proposition 2).

Proposition 6. *There exists a separating function $g: I \to I$ which is onto and monotone and $g(R) = R$.*

Let $g_n: I \to I$ be an arbitrary separating monotone surjection such that $g_n(R) = R$ and $g_n(0) = 0$. We consider a continuum I_∞ defined as the limit of the inverse sequence $\{I_n, h_n g_n\}$ where for each $n > 1$ we have $I_n = I$, $h_n g_n: I_{n+1} \to I_n$ and h_n are N-mappings from I onto I. Then we say that I_∞ is of type N^*. We will prove

Theorem 7. *Let Y be a proper subcontinuum of a hereditarily indecomposable continuum X and let W be an open subset of X such that $Y \cap \bar{W} = \emptyset$. If f is a continuous mapping from Y onto I_∞ of type N^*, then there is a continuous mapping f^* from X onto I_∞ such that $f^*|Y = f$ and $f(\bar{W}) = (0, 0, \ldots)$.*

Proof. We denote the composition of f and the projection from the inverse limit I_∞ onto I_n by π_n. If $n < m$, then we put $\alpha_n = h_n g_n$ and $\alpha_n^m = \alpha_n \alpha_{n+1} \cdots \alpha_{m-1}$. If $r \in R$ and $n < m$ then we define

$$A_n^m(r) = \pi_m^{-1}(\operatorname{int}(\alpha_n^m)^<(r)),$$

$$B_n^m(r) = \pi_m^{-1}(\operatorname{int}(\alpha_n^m)^>(r)),$$

$$A_n(r) = A_n^{n+1} \cup A_n^{n+2} \cup \cdots,$$

$$B_n(r) = B_n^{n+1} \cup B_n^{n+2} \cup \cdots.$$

It is clear that

(1) $A_n(r)$ and $B_n(r)$ are open F_σ-sets in X.

It suffices to construct a sequence of continuous mappings $f_n: X \to I_n$ which satisfy the conditions

(2) $f_n(\bar{W}) = 0$,

(3) f_n are separating,

(4) if $r \in R \setminus \{0, 1\}$, then $A_n(r) \subset \operatorname{int} f_n^<(r)$ and $B_n(r) \subset \operatorname{int} f_n^>(r)$,

(5) $f_n | Y = \pi_n$,

(6) $f_{n-1} = \alpha_{n-1} f_n$ for $n > 1$.

We proceed by induction. Firstly we will construct a continuous mapping $f_1: X \to I$ which satisfies conditions (2)–(5) for $n = 1$. The construction is based on the proof of Urysohn's Lemma. Let R be arranged into an infinite sequence $0, 1, r_1, r_2, \ldots$ For every number $r_i \in R \setminus \{0, 1\}$ we shall define open sets $V_i, U_i \subset X$ subject to the conditions

(7) $X \setminus U_i \subset V_j$ whenever $r_i < r_j$,

(8) $\bar{W} \cup \pi_1^-(r_i) \subset V_i$, $\pi_1^+(r_i) \subset U_i$, $A_1(r_i) \subset X \setminus \overline{U_i}$, $B_1(r_i) \subset X \setminus \overline{V_i}$ *and* $\overline{U_i} \cap \overline{V_i} = \emptyset$.

The sets V_i and U_i will be defined inductively. Since π_1 is separating, we find open sets G_1 and H_1 such that $\bar{W} \cup \pi_1^-(r_1) \subset G_1$, $\pi_1^+(r_1) \subset H_1$ and $\overline{G_1} \cap \overline{H_1} = \emptyset$. Since the sets $\bar{W} \cup \pi_1^-(r_1)$ and $B_1(r_1)$ are separated F_σ-sets, from Proposition 1 we infer that there exist open sets G_2, $H_2 \subset X$ such that $\bar{W} \cup \pi_1^-(r_1) \subset G_2$, $B_1(r_1) \subset H_2$ and $G_2 \cap H_2 = \emptyset$. Similarly, since the sets $\bar{W} \cup A_1(r_1)$ and $\pi_1^+(r_1)$ are separated F_σ-sets, we obtain open sets G_3, $H_3 \subset X$ such that $\bar{W} \cup A_1(r_1) \subset G_3$, $\pi_1^+(r_1) \subset H_3$ and $G_3 \cap H_3 = \emptyset$. Put $V_1 = G_1 \cap G_2 \cap G_3$ and $U_1 = H_1 \cap H_2 \cap H_3$. The sets V_1 and U_1 satisfy (8) for $i = 1$, because $\pi_1^-(r_1) \subset A_1(r_1)$ and $\pi_1^+(r_1) \subset B_1(r_1)$.

Assume that the sets V_i and U_i are already defined for $i \leq n$ and satisfy (7) and (8) for $i, j \leq n$. Let us denote by r_1 and r_m respectively those of the numbers $r_1, r_2, \ldots, r_n$ that are closest to r_{n+1} from the left and from the right. We have $X \setminus U_1 \subset V_m$. Since π_1 is separating, we find open sets G_1 and H_1 in X such that $\bar{W} \cup \pi_1^-(r_{n+1}) \cup \bar{V}_1 \subset G_1$, $\pi_1^+(r_{n+1}) \cup X \setminus V_m \subset H_1$ and $\overline{G_1} \cap \overline{H_1} = \emptyset$. Since the sets $\bar{W} \cup \pi_1^-(r_{n+1}) \cup \bar{V}_1$ and $B_1(r_{n+1}) \cup X \setminus V_m$ are separated F_σ-sets, we find open sets G_2, $H_2 \subset X$ such that $\bar{W} \cup \pi_1^-(r_{n+1}) \cup \overline{V_1} \subset G_2$, $B_1(r_{n+1}) \cup X \setminus V_m \subset H_2$ and $G_2 \cap H_2 = \emptyset$. Similarly, we obtain open sets G_3 and H_3 in X such that $\bar{W} \cup A_1(r_{n+1}) \cup \overline{V_1} \subset G_3$, $\pi_1^+(r_{n+1}) \cup X \setminus V_m \subset H_3$ and $G_3 \cap H_3 = \emptyset$. The sets $V_{n+1} = G_1 \cap G_2 \cap G_3$ and $U_{n+1} = H_1 \cap H_2 \cap H_3$ have the required properties.

Put $V = V_1 \cup V_2 \cup \cdots$. The function f_1 from X to I defined by the formula

$$f_1(x) = \begin{cases} \inf\{r_i : x \in V_i\} & \text{for } x \in V, \\ 1 & \text{for } x \in X \setminus V \end{cases}$$

is continuous and separating, because $f_1^-(r_j) = \bigcup\{V_i : r_i < r_j\} \subset V_j$, $f_1^+(r_j) = \bigcup\{X \setminus \overline{V_i} : r_i > r_j\} \subset U_j$. Since $f_1(X \setminus U_i) \subset [0, r_i]$ and $f_1(X \setminus V_i) \subset [r_i, 1]$, we obtain that f_1 satisfies (2)–(5) for $n = 1$ by (8).

Assume the functions $f_1, f_2, \ldots, f_n$ have been constructed in such way that (2)–(6) hold for $k \leq n$. Now, we will construct f_{n+1} by induction.

Let a, b, c, d be the rationals which describe h_{n+1}. Define $P \subset X$ and $Q \subset X$ by

$$P = \{x \in \overline{f_n^-(b)} : K(x, \overline{f_n^-(b)}) \cap \overline{f_n^-(d)} \neq \emptyset\},$$

$$Q = \{x \in \overline{f_n^+(d)} : K(x, \overline{f_n^+(d)}) \cap f_n^+(b) \neq \emptyset\}.$$

As in the proof of Lemma 3 in [1], p. 8 one can check that

(9) P and Q are closed and disjoint.

Moreover

(10) $\overline{(\pi_{n+1} g_n)^+(a)} \cap P = \emptyset$ and $\overline{(\pi_{n+1} g_n)^-(c)} \cap Q = \emptyset$.

We will show only the first equality (a parallel argument will show the second). Suppose that $x \in \overline{(\pi_{n+1} g_n)^+(a)} \cap P$ and $z \in K(x, \overline{f_n^-(b)}) \cap \overline{f_n^-(d)}$. Since X is hereditarily indecomposable we infer that $K(x, \overline{f_n^-(b)}) \subset Y$. Since $(\pi_{n+1} g_n)^>(a) \subset \operatorname{int} f_n^>(d) \subset X \setminus \overline{f_n^-(d)}$, we obtain that $\pi_{n+1} g_n(K(x, \overline{f_n^-(b)}) \cap [0, a) \neq \emptyset$. Therefore $\pi_{n+1} g_n(K(x, \overline{f_n^-(b)} \cap \operatorname{int} (\alpha_n^{n+1})^>(b) \neq \emptyset$, a contradiction.

We are obviously have

(11) $\overline{(\pi_{n+1} g_n)^-(a)} \subset \overline{f_n^-(b)}$ and $\overline{(\pi_{n+1} g_n)^+(c)} \subset \overline{f_n^+(d)}$.

Furthermore

(12) $(\pi_{n+1} g_n)^{-1}(a) \cap Q = \emptyset$ and $(\pi_{n+1} g_n)^{-1}(c) \cap P = \emptyset$.

As above, suppose that $x \in (\pi_{n+1} g_n)^{-1}(a) \cap Q$ and $z \in K(x, \overline{f_n^+(d)}) \cap \overline{f_n^+(b)} \neq \emptyset$. Since X is hereditarily indecomposable we conclude that $K(x, \overline{f_n^+(d)}) \subset Y$. Hence $\pi_{n+1} g_n(K(x, \overline{f_n^+(d)}) \cap \operatorname{int}(\alpha_n^{n+1})^{-1}[0, d] \neq \emptyset$. But $\pi_{n+1}^{-1}(\operatorname{int}(\alpha_n^{n+1})^{-1}[0, d]) \subset A_n(d) \subset \operatorname{int} f_n^<(d) \subset X \backslash \overline{f_n^+(d)}$, a contradiction.

The construction guarantees that

(13) every component of $\overline{f_n^-(b)}$ which intersects $\overline{(\pi_{n+1} g_n)^-(a)}$ is disjoint with the set $\overline{(\pi_{n+1} g_n)^+(a)} \cup Q$ and every component of $\overline{f_n^+(d)}$ which intersects $(\pi_{n+1} g_n)^+(c)$ is disjoint with the set $\overline{(\pi_{n+1} g_n)^-(c)} \cup P$.

It follows from (9), (13), and Lemma 2 in [1], p. 7 that there is a separation $A \cup M$ of $\overline{f_n^-(b)}$ such that $P \cup \overline{(\pi_{n+1} g_n)^-(a)} \subset A$ and $Q \cup \overline{(\pi_{n+1} g_n)^+(a)} \cap \overline{f_n^-(b)} \subset M$. Similarly, there is also a separation $B \cup N$ of $\overline{f_n^+(d)}$ such that $Q \cup \overline{(\pi_{n+1} g_n)^+(c)} \subset B$ and $(A \cup \overline{(\pi_{n+1} g_n)^-(c)}) \cap \overline{f_n^+(d)} \subset N$. Then

(14) A and B are disjoint.

Moreover, as in the proof of Lemma 3 in [1], p. 9 we have

(15) $X = A \cup B \cup (M \cap N) \cup (f_n^{-1}(d) \cap M) \cup (f_n^{-1}(b) \cap N)$.

Put $J = A \cup (f_n^{-1}(b) \cap N)$, $K = (f_n^{-1}(b) \cap N) \cup (M \cap N) \cup (f_n^{-1}(d) \cap M)$ and $L = (f_n^{-1}(d) \cap M) \cup B$ and define β by

$$\beta(x) = \begin{cases} (h_n|[0, a])^{-1}(f_n(x)) & \text{for } x \in J, \\ (h_n|[a, c])^{-1}(f_n(x)) & \text{for } x \in K, \\ (h_n|[c, 1])^{-1}(f_n(x)) & \text{for } x \in L. \end{cases}$$

To construct f_{n+1} we now proceed as follows. For every number $r_i \in R$ we define sets V_i, $U_i \subset X$ satisfying conditions

(16) $X \backslash U_i \subset V_j$ whenever $r_i < r_j$,

(17) $\bar{W} \cup \pi_{n+1}^{-1}(r_i) \cup \beta^-(g_n(r_i)) \subset V_i$, $\pi_{n+1}^+(r_i) \cup \beta^+(g_n(r_i)) \subset U_i$, $A_{n+1}(r_i) \subset X \backslash \overline{U_i}$, $B_{n+1}(r_i) \subset X \backslash \overline{V_i}$ and $\overline{U_i} \cap \overline{V_i} = \emptyset$.

Next putting $V = V_1 \cup V_2 \cdots$ we define f_{n+1} by

$$f_{n+1}(x) = \begin{cases} \inf\{r_i : x \in V_i\} & \text{for } x \in V, \\ 1 & \text{for } x \in X \backslash V. \end{cases}$$

Since this construction is similar to the construction of f_1, we omit the details. The proof of Theorem 7 is complete.

Since a pseudoarc is of type N^* (see [4], Corollary 17) we have the following.

Corollary 8. *If Y is a proper subcontinuum of a hereditarily indecomposable continuum X, W is an open subset of X such that $Y \cap \bar{W} = \emptyset$, and f is a continuous surjection from X onto a pseudoarc P, then there is a continuous mapping f from X onto P such that $f | Y = f$ and $f(\bar{W}) = (0, 0, \ldots)$.*

Corollary 9. *A pseudoarc is a retract of every hereditarily indecomposable continuum containing it.*

The above corollaries generalize results from [4] to the nonmetric case. Moreover, we conclude also the following.

Corollary 10. *If* $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$ *are fixed n-point sets in a pseudoarc P, then there is a continuous surjection f from P onto P such that* $f(a_i) = b_i$.

In fact, let A_i be small pseudoarc such that $a_i \in A_i$ and the A_i's are pairwise disjoint. It follows from [1] that there are continuous surjections f_i from A_i onto P. Since P is homogeneous there is a homeomorphism h_i from P onto P such that $h_i(f_i(a_i)) = b_i$. Take open sets W_i such that $A_i \subset \bar{W}_i \subset X \setminus \bigcup_{j \neq i} A_j$. By Corollary 8 there is a continuous mapping g_i from P onto P such that $g_i | A_i = h_i f_i$ and $g_i(X \setminus \bar{W}_i) = 0$. Then the mapping g from P onto P defined by $g(x) = g_i(x)$ for $x \in W_i$ and $g(x) = 0$ for $x \in X \setminus (W_1 \cup \cdots \cup W_n)$ has required properties.

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